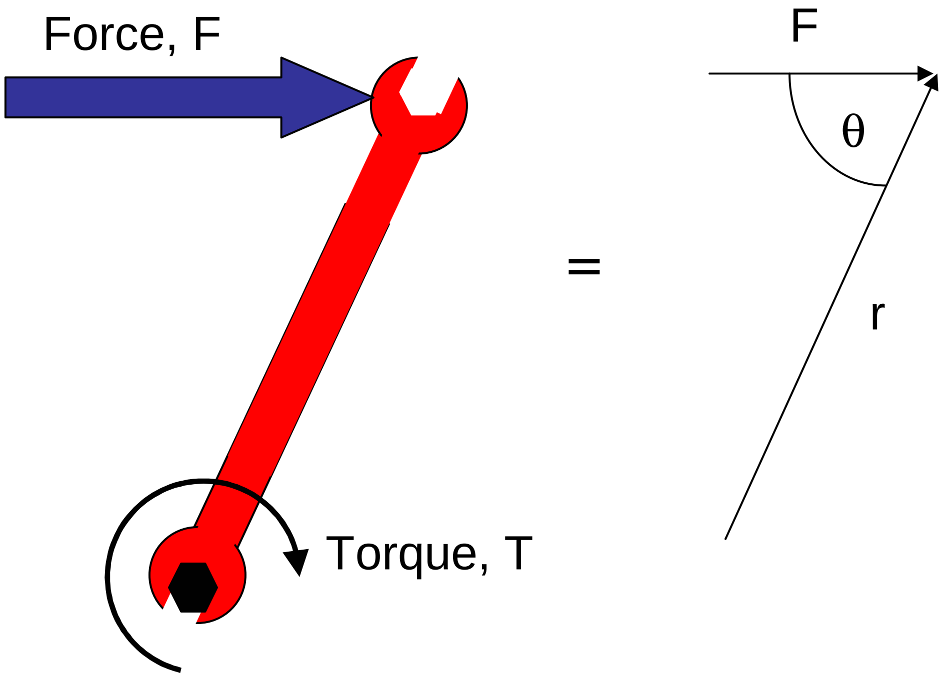
Area Under the Torque vs. RPM Curve: Average Power

Some Basics

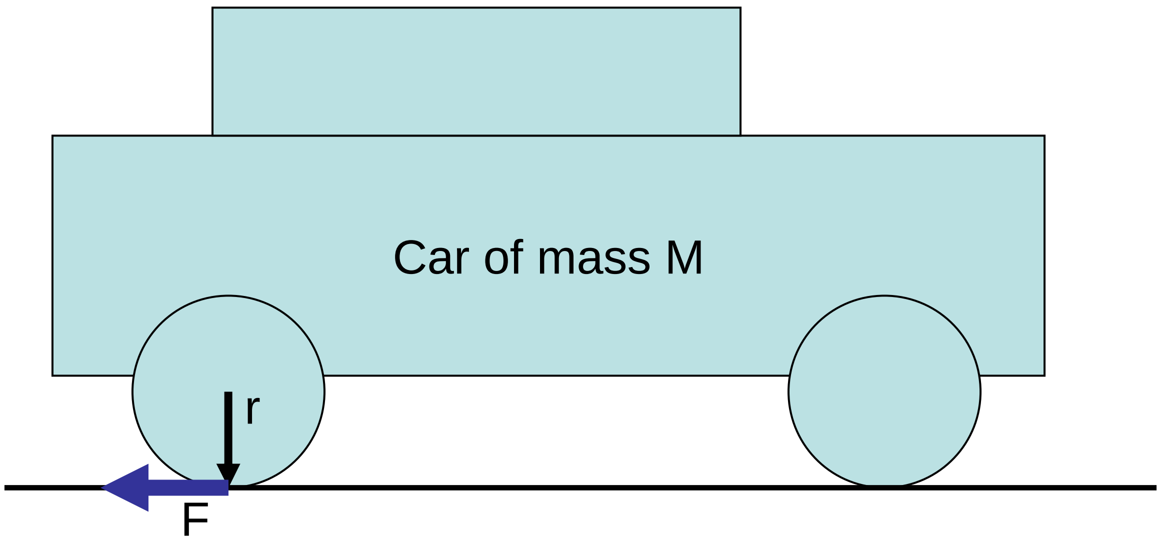
- What is torque?
  - Consider a wrench on a nut, the torque about the nut is



$$T = rF\sin(\theta)$$

  - If F is at a right angle to moment arm r then T=rF

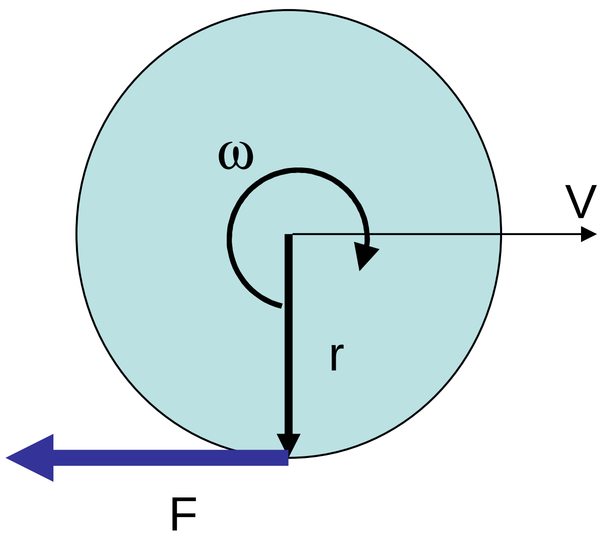
How does the Tire apply torque to the road?



F=Ma, a is the acceleration of the car

Then

T=rF=rMa



The linear velocity of center of the wheel (and the car's velocity) Is given by $V = r\omega$

Where ω is the angular or rotational speed of the wheel in radians/sec or

$\omega = 2 * \pi * RPM/60$

Now for a bit of Calculus and basic Formulas

$$a = \frac{dV}{dt} \qquad \text{eqn.1}$$

$$\text{Using v} = \text{r}\omega$$

$$a = r\frac{d\omega}{dt} \qquad \text{eqn. 2}$$

$$T = rMa \qquad \text{eqn. 3}$$

$$V = r\omega \qquad \text{eqn. 4}$$

$$P = FV \qquad \text{eqn. 5}$$

Where:

a = acceleration

T = Torque

V = velocity

r = radius of wheel

ω = angular speed of wheel

P = power, Force times velocity

Average Power Over a Given Time Interval

$$\langle P \rangle = \frac{\int_{t1}^{t2} P\,dt}{\int_{t1}^{t2} dt}$$

I will later relate this to average Acceleration and final velocity

It would be useful to have the average power equation in terms of the corresponding Shift points or ω1 and ω2. Then we could relate a dyno curve (Torque vs. ω or RPM) to the average power & acceleration. Then:

Combining eqns 2 and 3 gives:

$$\frac{\mathrm{d}\omega}{\mathrm{dt}} = \frac{T}{r^2 M}$$

Average Power in Terms of ω Or RPM

$$\langle P \rangle = \frac{\displaystyle\int_{\omega 1}^{\omega 2} \frac{P}{\frac{d\omega}{dt}} d\omega}{\displaystyle\int_{\omega 1}^{\omega 2} \frac{1}{\frac{d\omega}{dt}} d\omega}$$

Now recall that:

$P = FV = Fr\omega$

And:

$F = Ma$

Then:

$$P = Mar\omega, \qquad a = r\frac{d\omega}{dt}, \quad P = Mr^2\frac{d\omega}{dt}\omega$$

Using $\quad T = rMa$

$$T = r^2 M\frac{d\omega}{dt}, \qquad \frac{P}{\frac{d\omega}{dt}} = Mr^2\omega, \quad \frac{1}{\frac{d\omega}{dt}} = \frac{Mr^2}{T}$$

The final result is Then

$$\langle P \rangle = \frac{\displaystyle\int_{\omega 1}^{\omega 2} \frac{P}{\frac{d\omega}{dt}}\, d\omega}{\displaystyle\int_{\omega 1}^{\omega 2} \frac{1}{\frac{d\omega}{dt}}\, d\omega}, \qquad \langle P \rangle = \frac{\displaystyle\int_{\omega 1}^{\omega 2} \omega d\omega}{\displaystyle\int_{\omega 1}^{\omega 2} \frac{1}{T}\, d\omega}$$

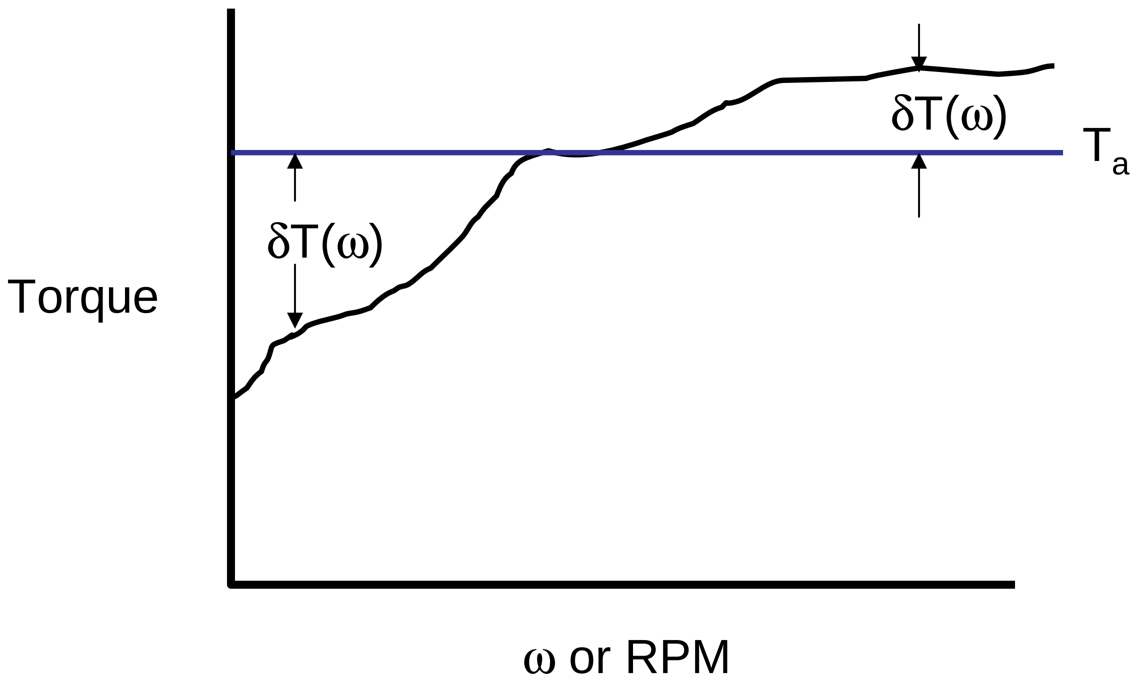
Now for an assumption-
Assume that the torque is given by some average torque and a small term which varies with ω or RPM
(I will later show that this assumption is not that restrictive):

$$T(\omega) = T_a + \delta T(\omega)$$

Where:

$$T_a = \frac{\int_{\omega 1}^{\omega 2} T d\omega}{\int_{\omega 1}^{\omega 2} d\omega}, \text{ and } \quad \frac{\delta T(\omega)}{T_a} << 1$$

Graphically This Is



$$\langle P \rangle = \frac{\int_{\omega 1}^{\omega 2} \omega d\omega}{\int_{\omega 1}^{\omega 2} \frac{1}{T} d\omega}$$

To examine this term….

$$\int_{\omega 1}^{\omega 2} \frac{1}{T} d\omega = \int_{\omega 1}^{\omega 2} \frac{1}{T_a + \delta T(\omega)} d\omega = \frac{1}{T_a} \int_{\omega 1}^{\omega 2} \frac{1}{1 + \frac{\delta T(\omega)}{T_a}} d\omega$$

$$\frac{1}{T_a} \int_{\omega 1}^{\omega 2} \frac{1}{1 + \frac{\delta T(\omega)}{T_a}} d\omega \cong \frac{1}{T_a} \int_{\omega 1}^{\omega 2} \left(1 - \frac{\delta T(\omega)}{T_a} + \left(\frac{\delta T(\omega)}{T_a} \right)^3 - \ldots. \right) d\omega$$

Where the series expansion is used for $\frac{1}{1+x}$ when x is small:

$$\frac{1}{1+x} \cong 1 - x + x^2 - x^3 + \ldots\ldots..$$

And the $\cong$ symbol means "approximately" or "close enough"

Because $\dfrac{\delta T(\omega)}{T_a}$ is small, then $\left(\dfrac{\delta T(\omega)}{T_a}\right)^2$ is very small

that is a small number squared is very small - think about it $.1^2 = .01$

I will retain terms only larger than $\left(\dfrac{\delta T(\omega)}{T_a}\right)^2$

$$\frac{1}{T_a}\int_{\omega 1}^{\omega 2}\frac{1}{1+\dfrac{\delta T(\omega)}{T_a}}d\omega \cong \frac{1}{T_a}\int_{\omega 1}^{\omega 2}\left(1-\frac{\delta T(\omega)}{T_a}\right)d\omega$$

$$\langle P\rangle = \frac{\int_{\omega 1}^{\omega 2}\omega d\omega}{\int_{\omega 1}^{\omega 2}\frac{1}{T}d\omega} \cong \frac{\int_{\omega 1}^{\omega 2}\omega d\omega}{\frac{1}{T_a}\int_{\omega 1}^{\omega 2}\left(1-\frac{\delta T(\omega)}{T_a}\right)d\omega} = \frac{T_a\int_{\omega 1}^{\omega 2}\omega d\omega}{\int_{\omega 1}^{\omega 2}d\omega-\int_{\omega 1}^{\omega 2}\frac{\delta T(\omega)}{T_a}d\omega}$$

$$= \frac{T_a\int_{\omega 1}^{\omega 2}\omega d\omega}{\int_{\omega 1}^{\omega 2}d\omega\left(1-\frac{\int_{\omega 1}^{\omega 2}\frac{\delta T(\omega)}{T_a}d\omega}{\int_{\omega 1}^{\omega 2}d\omega}\right)} \cong \frac{T_a\int_{\omega 1}^{\omega 2}\omega d\omega}{\int_{\omega 1}^{\omega 2}d\omega}\left(1+\frac{\int_{\omega 1}^{\omega 2}\frac{\delta T(\omega)}{T_a}d\omega}{\int_{\omega 1}^{\omega 2}d\omega}\right)$$

Again using the series expansion for $\frac{1}{1+\mathrm{x}}$ or $\frac{1}{1-\mathrm{x}}$ for small x

$$\langle P \rangle \cong \frac{T_a \int_{\omega 1}^{\omega 2} \omega d\omega}{\int_{\omega 1}^{\omega 2} d\omega} \left(1 + \frac{\int_{\omega 1}^{\omega 2} \frac{\delta T(\omega)}{T_a} d\omega}{\int_{\omega 1}^{\omega 2} d\omega} \right)$$

$$= \frac{\int_{\omega 1}^{\omega 2} \omega d\omega}{\int_{\omega 1}^{\omega 2} d\omega} \left(\frac{\int_{\omega 1}^{\omega 2} T_a + \delta T(\omega) d\omega}{\int_{\omega 1}^{\omega 2} d\omega} \right) = \frac{\int_{\omega 1}^{\omega 2} \omega d\omega}{\int_{\omega 1}^{\omega 2} d\omega} \left(\frac{\int_{\omega 1}^{\omega 2} T d\omega}{\int_{\omega 1}^{\omega 2} d\omega} \right)$$ Recalling that $T = T_a + T(\omega)$

Now $\int_{\omega 1}^{\omega 2} \omega d\omega = \frac{\omega 2 + \omega 1}{2} \int_{\omega 1}^{\omega 2} d\omega = \varpi \int_{\omega 1}^{\omega 2} d\omega$

Where ϖ is the average of $\omega 2 + \omega 1$ or $\frac{\omega 2 + \omega 1}{2}$

$$\langle P \rangle \cong \varpi \left(\frac{\int_{\omega 1}^{\omega 2} T d\omega}{\int_{\omega 1}^{\omega 2} d\omega} \right)$$ or $$\langle P \rangle \cong \frac{\varpi}{\omega 2 - \omega 1} \int_{\omega 1}^{\omega 2} T d\omega$$

This is the desired result, the average power over a given time interval is $\langle P \rangle \cong \text{Constant} * \int_{\omega 1}^{\omega 2} T d\omega$

because $\frac{\varpi}{\omega 2 - \omega 1}$ is a constant once the shift points are determined, say 2000RPM and 6000 RPM

$$\langle P \rangle \cong \frac{\varpi}{\omega 2 - \omega 1} \int_{\omega 1}^{\omega 2} T d\omega$$

To get an expression for the average acceleration, $\langle a \rangle$ note that :

$$F = Ma, \quad T = rF = rMa, \quad \text{then} \quad a = \frac{T}{rM}$$

$$\langle a \rangle = \frac{\int_{\omega 1}^{\omega 2} T d\omega}{rM \int_{\omega 1}^{\omega 2} d\omega}$$

This shows that the average acceleration over a time interval is $\langle a \rangle = \text{Constant} * \int_{\omega 1}^{\omega 2} T d\omega$

So for a given car with a given mass, wheel geometry and fixed shift points, the average Power and the average acceleration is proportional to the AREA UNDER THE TORQUE CURVE.

Maximize the area under the torque vs. RPM curve and you Will maximize acceleration and final speed.

POSITIVE DISPLACEMENT vs. CENTRIFIGAL BLOWERS

- It is clear from the graphical example that positive displacement blowers have a very small $\delta T(\omega)$ term because they tend to produce a relatively flat torque curve over the RPM range.
- However, it is not clear that for Centrifugals having a torque curve building linearly with RPM that the preceding derivation is valid….So lets check it with an example- (*Hint: This is where I will show that the assumption that torque is given by some average torque and a small term which varies with w or RPM*) is not restrictive and the **Area Under the torque rule curve applies even for the Centrifugals**

An Example Centrifugal Torque Curve

- This example is taken from real dyno data posted on modularfords.com, probably a mustang GT (from the cutoff RPM of 6000)

Fitted Torque function:

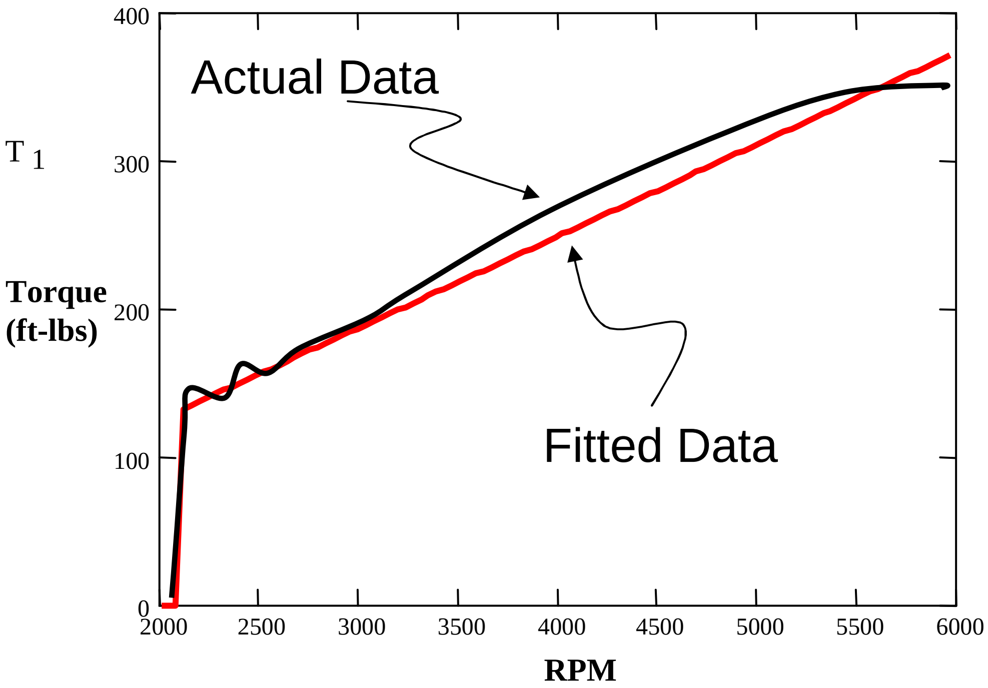
$$T(RPM) := \frac{T_2 - T_1}{RPM_2 - RPM_1} \cdot \left(RPM - RPM_1\right) + T_1$$

$RPM_1 := 2000$ $RPM_2 := 6000$

$T_1 := 125$ $T_2 := 373$

Ratio of T_2/T_1=2.98

R≈3



- With an exact equation, I can calculate the difference between the exact expression for average power and the approximate equation based on the area under the toque curve:

$$\langle P \rangle Approx = \frac{\varpi}{\omega 2 - \omega 1} \int_{\omega 1}^{\omega 2} T d\omega$$

$$\langle P \rangle Exact = \frac{\varpi(\omega 2 - \omega 1)}{\int_{\omega 1}^{\omega 2} \frac{1}{T} d\omega}$$

Recalling that :

$$\omega = \frac{RPM}{60} 2\pi$$

$$\varpi = \frac{\omega_2 + \omega_1}{2}$$

$$\langle P \rangle Approx = \overline{RPM} \frac{T_2 + T_1}{2}$$

$$\langle P \rangle Exact = \overline{RPM} \frac{T_2 - T_1}{\ln\left(\frac{T_2}{T_1}\right)}$$

$$\overline{RPM} = \frac{RPM_2 + RPM_1}{2}$$

$$\langle P \rangle Approx = \overline{RPM}\,\frac{T_2 + T_1}{2} \qquad \langle P \rangle Exact = \overline{RPM}\,\frac{T_2 - T_1}{\ln\left(\frac{T_2}{T_1}\right)}$$

Using $R = \frac{T_2}{T_1}$ gives:

$$\langle P \rangle Approx = T_1\overline{RPM}\,\frac{R+1}{2} \qquad \langle P \rangle Exact = T_1\overline{RPM}\,\frac{R-1}{\ln(R)}$$

- These do not look like the same thing but are they close?....YES! Need to show (R+1)/2 ~(R-1)/ln(R) for reasonable values of R (say R=1-4

- Need to look for a series expansion for (R-1)/ln(R):

$$\frac{\mathrm{R}-1}{\ln(\mathrm{R})} := 1 + \frac{1}{2}\cdot(\mathrm{R}-1) - \frac{1}{12}\cdot(\mathrm{R}-1)^2 + \frac{1}{24}\cdot(\mathrm{R}-1)^3 - \frac{19}{720}\cdot(\mathrm{R}-1)^4$$

- For reasonable values of R this can be approximated by:

$$\frac{R-1}{\ln(R)} = 1 + \frac{1}{2}(R-1) = \frac{R+1}{2}$$

- So that (R+1)/2 ~(R-1)/ln(R) for reasonable values of R

- Same result! Exact and approximate answers are very close ~15% Even for torque ratios of 4! At torque ratios of 373/125=3 they are 10% different but the approximate gives the higher answer (more area under the torque curve)

- Bottom line- **Area Under the torque rule curve applies even for the Centrifugals**